Ho and Stoll (1978-1981), The Dealer Problem - Introduction

A model on Market Making and inventory control.

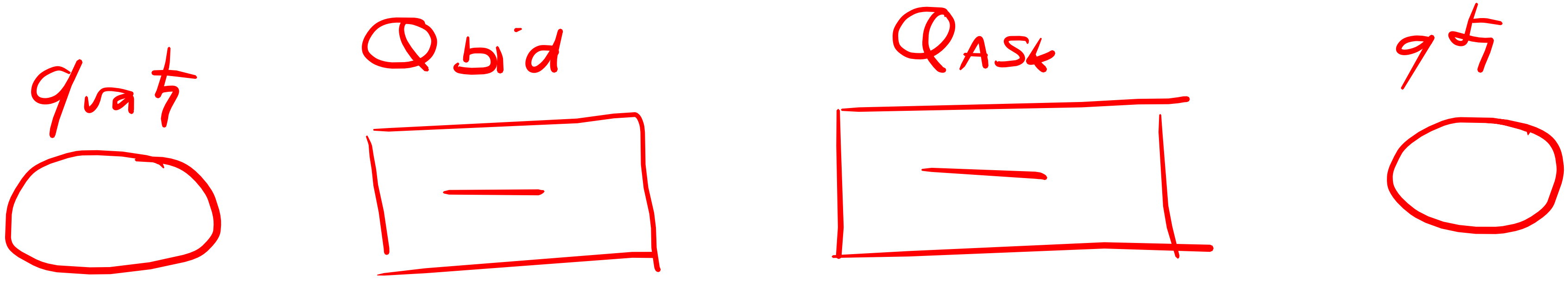
References

- Optimal dealer pricing under transactions and return uncertainty, Journal of Financial Economics (Ho and Stoll, 1981)
- High-frequency trading in a limit order book (Avellaneda and Stoikov, 2008)

Key components

- pricing/modeling based on public information (objective, public)
- inventory control/risk management (subjective, private)

Different bid/ask prices are posted by different market making institutions (dealers), taking into account their own portfolio (inventory), risk aversion, time horizon and hedging capabilities. Avellaneda and Stoikov add execution optimization.


qvat
Qbid
QASK
qt

The Dealer problem - Just quoting the original paper

$\boxed{r_b}$ $\quad p \quad$ $\boxed{r_a}$

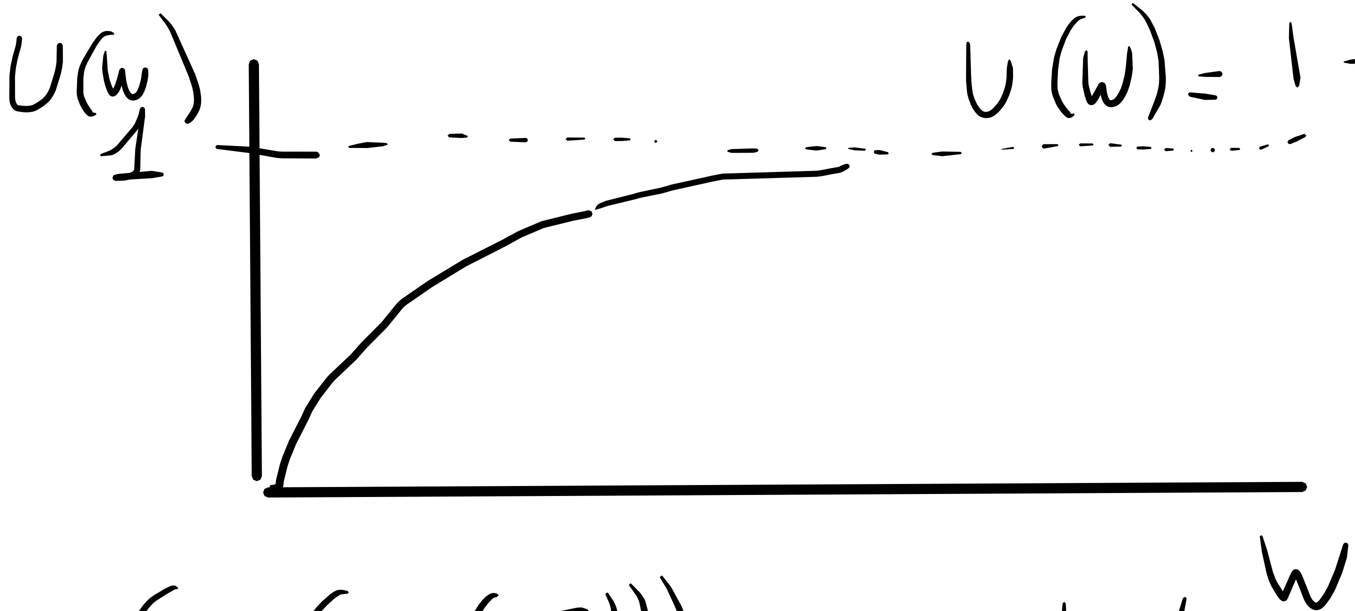
$p-b \quad\quad\quad\quad p+a$

The solution of the problem gives an optimal reservation selling fee, a; and an optimal reservation buying fee, b. The *reservation fee* is the minimum fee such that the dealers expected utility of terminal wealth would not be lowered were he to trade at that fee. If p is the true price of the stock in the opinion of the dealer, the dealer would earn the fee by buying at $p - b = r_b$, the bid price, and selling at $p + a = r_a$, the ask price.

$W(T) \longrightarrow U(W(T))$

health

$u(w) = 1 - e^{-\gamma w}$



$\underline{E}(\underline{U}(W(T))) \rightarrow$ constant

→ NOTHING HAPPEN

→ TRADE

$$S(T) \sim N(s, \sigma_s^2)$$

$$W(T) = \underline{w_0} + q \cdot S(T)$$

$$E(U(T)) = E(\underbrace{1 - e^{-\gamma W(T)}}_{U})$$

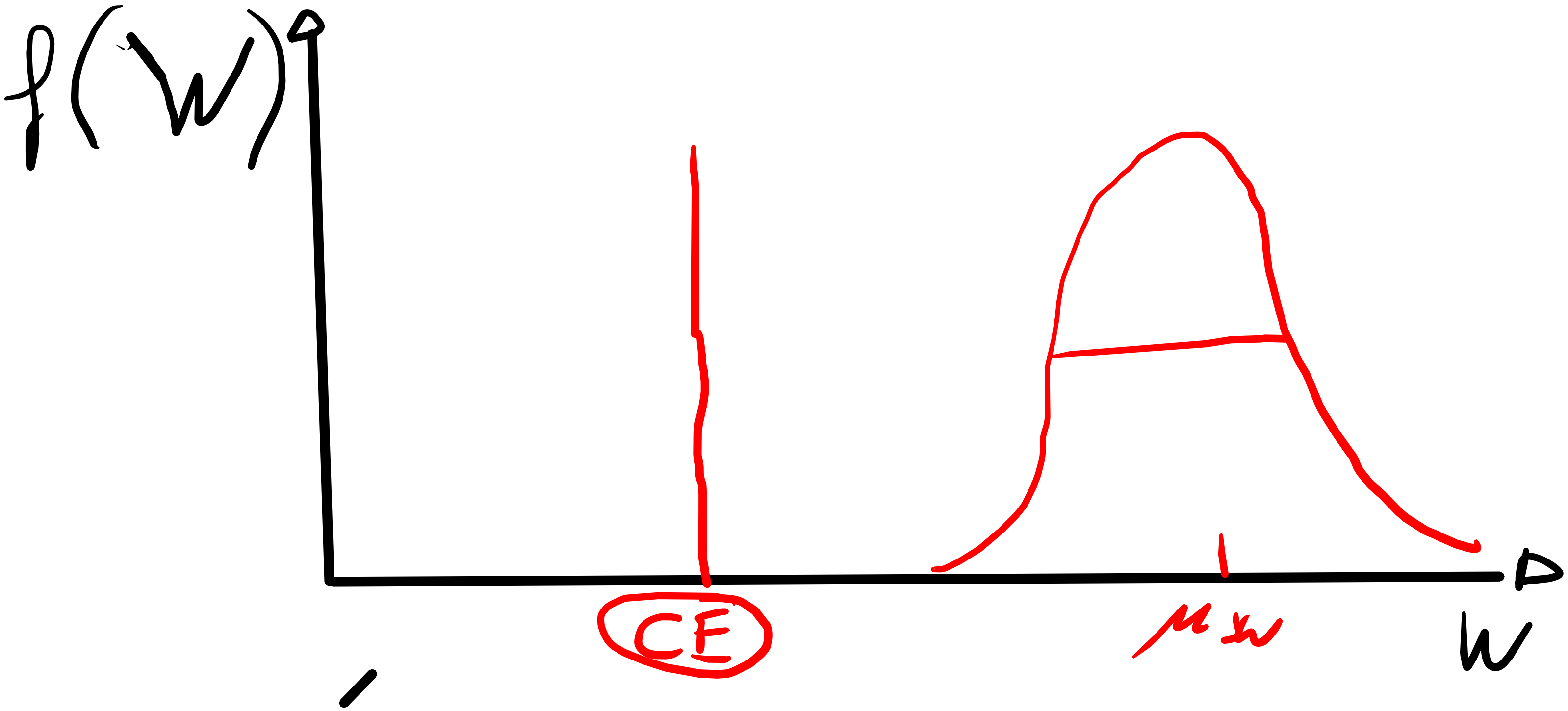
$$W(T) \sim N(w_0 + q \cdot s,\ q^2 \sigma_s^2)$$

$$E(U(W(T))) \sim 1 - e^{-\gamma \underbrace{(w_0 + qs - \frac{1}{2}\gamma q^2 \sigma_s^2)}_{\text{mean} - \frac{1}{2}\gamma \cdot \text{variance}}}$$

$$W(T) \sim N(\mu_w, \sigma_w^2)$$

$$E(U(W(T))) = 1 - e^{-\gamma(\mu_w - \frac{1}{2}\gamma \sigma_w^2)}$$

$$\underbrace{\left[W_0 - sq \qquad q^2 \sigma_s^2\right]}_{C.E.}$$



1° CASE NO TRADE

(WE START WITH q shares $\downarrow$ q shares at time T)

C.E. $\cancel{\mu_w - \frac{1}{2}\gamma\sigma_w^2}$

$w_0 + q \cdot s - \frac{1}{2}\gamma q^2 \sigma_s^2$

2° CASE TRADE OCCURS at r_b and buy $\underline{\underline{dq}}$ of shares

C.E. $w_0 + (q+dq) \cdot s - \frac{1}{2}\gamma(q+dq)^2 \cdot \sigma_s^2 - dq \cdot r_b$

$$\cancel{w_0} + \cancel{q s} - \frac{1}{2}\cancel{\gamma q^2 \sigma_s^2} = \cancel{w_0} + \cancel{q s} + dq \cdot s - \frac{1}{2}\gamma(\cancel{q^2} + 2dq \cdot q + dq^2)\sigma_s^2$$

$$0 = \cancel{dq} \cdot s - \gamma \cancel{dq} q \sigma^2 - \frac{1}{2}\gamma dq^{\cancel{2}} \sigma_s^2 - \cancel{dq} \cdot r_b$$

$$r_b = s - \gamma q \sigma_s^2 - \frac{1}{2}\gamma\sigma_s^2 dq$$

The Dealer Problem - Solution

Main idea: price indifference taking into account risk aversion. No superior forecasting capabilities are required.
Given a security S (or whatever risk factor) assume that conditional on all available information (public and private, including your trades)

$$S(T) \sim N(s, \sigma^2 T) \tag{8}$$

introduce a risk aversion model with utility function (C.A.R.A., Constant Absolute Risk Aversion)

$$U(W) = 1 - e^{-\gamma W} \tag{9}$$

The Dealer Problem - Solution

$S \; \sigma_S^2 \quad \sigma^2 \cdot T$ $\qquad$ $\sigma_S^2 = \sigma^2 \cdot T$

Let's assume that the dealer's initial inventory (i.e. portfolio) is represented by q units of security S plus initial wealth W_0, and dq is the quoted quantity. Compute $E(U(W(S(T))))$ where $W(T) = W_0 + qS(T)$ and use the Certainty Equivalent defined as $CE = U^{-1}E[U(W(S(T)))]$.
Let $\Gamma = \gamma\sigma^2 T$.
Requiring the same CE, independently of the quote being executed, it can be shown that the reservation price bid r^b and ask r^a are given by

$$\boxed{r^b = s - \Gamma q - (\Gamma/2)dq} - c \tag{10}$$

$$r^a = s - \Gamma q + (\Gamma/2)dq + c \tag{11}$$

where c must be positive to compensate for the operational costs.

$$r_b = S - q\gamma\sigma^2 T - \frac{1}{2}\gamma\, dq\, \sigma^2 T$$

Fair price

$$= S - \Gamma q - \frac{1}{2}\Gamma\, dq - c$$

$\boxed{q} \to s$

$\boxed{v_{ask}}$

$\uparrow + \frac{1}{2}\Gamma dq$

$s - \Gamma \cdot q$

$\downarrow - \frac{1}{2}\Gamma dq$

$\boxed{v_{ask}}$

$\boxed{v_{bid}}$

$s - \Gamma \cdot (q + dq)$

$q \to \boxed{q+dq}$

$\boxed{v_{bid}}$

$$\Gamma = \gamma \sigma^2 T$$

Market Maker pseudo-code

SimpleQuotingStrategy.m

Predictability vs. unpredictability and profitability

NO TRADE

TRADE

$$S(T) \sim \underbrace{N(\mu_s, \sigma_s^2)}$$

In the Ho and Stoll model a dealer earns a profit even if the underlying follows a random walk(he/she earns the spread). One of the main assumptions was

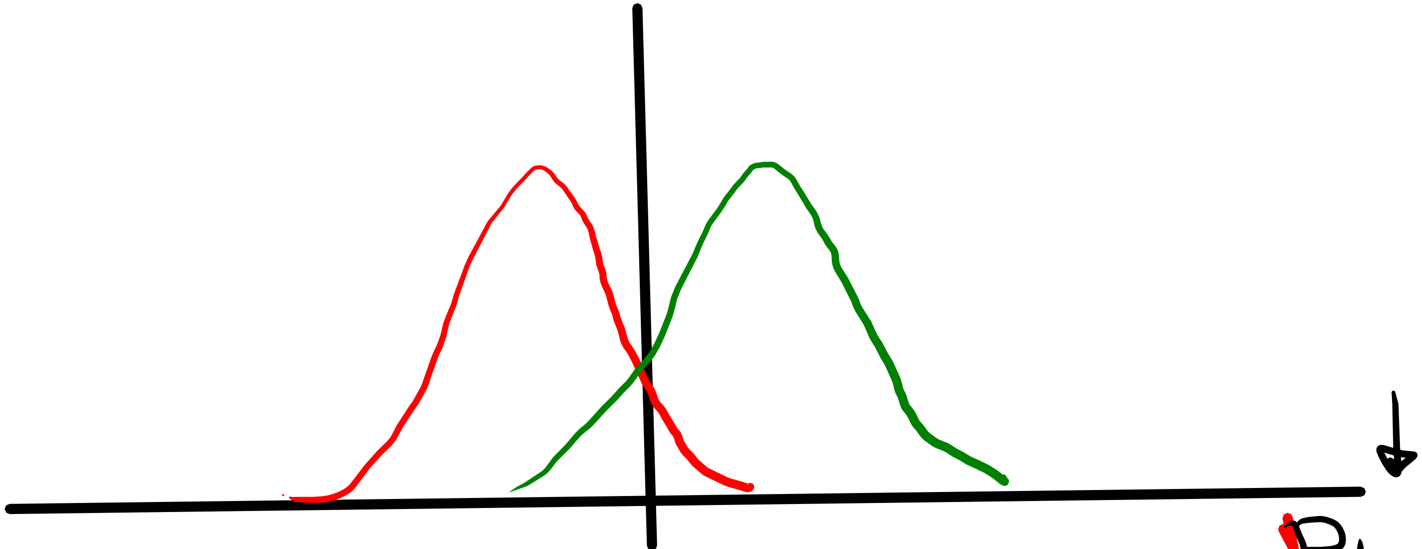
$$\underline{pdf^t(S_T}|\text{own trades at time t}) \sim N(s, \sigma_s^2) \tag{12}$$

i.e. the terminal price distribution of S doesn't change after a quote is executed. In general, the expected return of a non anticipative trading strategy is zero, assuming no spread and no fees, since its P&L can be modeled using a stochastic integral with zero expected value.

OWN TRADES

$$t \longrightarrow P^{t}_{BUY} \longrightarrow \underbrace{P_{t+\Delta t}}_{0.5\,(\text{bestbid} + \text{bestask})}$$

$$P^{t}_{sell} \longrightarrow P_{t+\Delta t}$$



$P_{t+\Delta t} - P_t$

$$S(T) \sim N(s, \sigma_s^2)$$

Noise and P&L distribution

Consider the following strategy: given a security S with Brownian-like prices, such that

$$dS_t = \sigma dB_t \tag{13}$$

hold a position $\sigma^{-1}B_t$ at each point in time. The P&L G_T at terminal time Tcan be written as follows (no fees, spread, etc..)

$$G_T = \int_0^T B_t dB_t \tag{14}$$

It can be shown that while $E(G_T) = 0$, the pdf of G_T is highly skewed. So this simple model shows that an algorithmic strategy with zero expected return in an efficient market may have a significant effect on the risk profile of the resulting P&L.